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Results on chromatic sum of graphs

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Abstract

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The *chromatic sum* of a graph G is the smallest sum of colors on the vertices among all the proper colorings with natural numbers. The *strength* of G is the number of colors that are used to attain the chromatic sum. In this thesis, we study some constructions of trees which demonstrate that the number of colors required to achieve this chromatic sum may be far from trivial. We then extend this result to two other constructions of general graphs that are not trees. We also compare and contrast these classes of graphs in terms of their orders and strengths.

Acknowledgements

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Background

Introduction

Graph coloring is an area of Graph Theory that has received much attention over the years. Its prominence is undoubtedly due to its involvement with the Four Color Problem, which is easy to state and understand, but the proof of which remained unknown from 1852 to 1976. The Four Color Theorem states that any map that can be drawn on the surface of a sphere can be colored with four colors in such a way that each country has exactly one color and any two neighboring countries have different colors. We notice that with any map we may associate a planar graph whose vertices correspond to countries and edges join two vertices if the corresponding countries share a common border.

A coloring of a graph is an assignment of colors to the vertices so that adjacent vertices have different colors. The minimum number of colors is given a special name – the chromatic number.

In this thesis we study a new variation of the chromatic number of a graph. Firstly, we use natural numbers 1, 2, … instead of colors blue, red, … on our graphs. Secondly, instead of minimizing the numbers of colors in a coloring, we minimize the sum of these colors over all vertices and call this sum the chromatic sum.

At first glance it may seem that the chromatic sum should be achieved by the same number of colors as the chromatic number. But it is a surprising fact that there are numerous examples of graphs whose chromatic sum is achieved by using a much larger number of colors than the minimum (chromatic) number. A simple example follows.

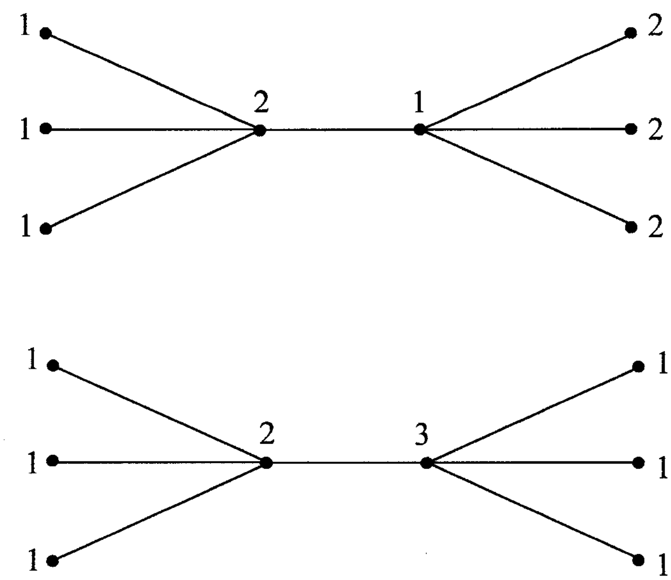

Figure 1.1. Example of trees.

In Figure 1.1 above, we notice that the sum of colors is 12 when we use two colors, but the sum of colors is reduced to 11 by adding a color 3. From the example above, it can be seen that although trees require only two colors to color them properly, their chromatic sum can be achieved by using additional colors. In this study we shall show that this result not only holds for trees but also for general graphs.

A generalization of the chromatic sum is the cost-chromatic number of a graph G first studied by Supowit [8]. Instead of limiting the colors to just natural numbers, he extends these to any positive rational number and calls them costs. This will not be studied in this thesis and we are not going to consider it any further. The interested reader can refer to [6], [7], and [8].

Terms and Definitions

We now formally introduce many of the basic terms, notations, and definitions that will be used in this thesis. We shall define other more specialized terms as needed in the body of the text.

A ***graph*** G is a finite nonempty set of objects called ***vertices,*** together with a (possibly empty) set of unordered pairs of distinct vertices of G called ***edges***. The vertex set is denoted by V(G), while the edge set is denoted by E(G). The edge uv is said to ***join*** the vertices u and v. If e = uv is an edge of a graph G, then u and v are ***adjacent*** vertices, while u and e are ***incident***, as are v and e. It is customary to define or describe a graph by means of a diagram in which each vertex is represented by a point and each edge e = uv is represented by a line segment joining the points corresponding to u and v. A graph H is a ***subgraph*** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

The cardinality of the vertex set of a graph G is called the ***order*** of G, and is commonly denoted by $|G|$. The ***degree*** of a vertex v in a graph G is the number of edges of G incident with v, and is denoted by $d_G(v)$. The ***maximum*** degree of G is the maximum degree among all the vertices of G and is denoted by $\Delta(G)$.

A graph G_1 is ***isomorphic*** to another graph G_2 if there exists a one-to-one mapping ϕ, called an ***isomorphism***, from $V(G_1)$ onto $V(G_2)$ such that ϕ preserves adjacency, that is, $uv \in E(G_1)$ if and only if $\phi u \phi v \in E(G_2)$.

A graph is said to be ***complete*** if every two of its vertices are adjacent. Such a graph of order k is denoted by K_k. A graph is said to be ***bipartite*** if its vertex set V can be decomposed into two disjoint subsets V_1 and V_2 such that every edge in G joins a vertex in V_1 with a vertex in V_2.

A ***u-v walk*** of G is a finite, alternating sequence of vertices and edges $u = u_0, e_1, u_1, e_2, \ldots, u_{k-1}, e_k, u_k = v$, beginning with vertex u and ending with vertex v, such that $e_i = u_{i-1}u_i$ for i = 1, 2, 3, …, k. A ***u-v trail*** is a u-v walk in which no edge is repeated while a ***u-v path*** is a u-v walk in which no vertex is repeated. A nontrivial closed trail of a graph G is referred to as a ***circuit*** of G, and a circuit whose vertices are all distinct (of course except the first and the last vertices) is called a ***cycle***. An ***acyclic graph*** has no cycles.

A vertex u is said to be ***connected*** to a vertex v in a graph G if there exists a u-v path in G. A graph G is ***connected*** if every two of its vertices are connected. A graph that is not connected is called a ***disconnected graph***. A ***component*** of G is a connected subgraph of G not properly contained in any other connected subgraph of G.

Among the connected graphs, the simplest yet very important, are the trees. A ***tree*** is an acyclic connected graph, while a ***forest*** is an acyclic graph. Thus every component of a forest is a tree. A ***bridge*** of a graph G is an edge which when removed from G leaves the graph disconnected. We observe that a graph is a tree if and only if each of its edges is a bridge.

Let u_1 and u_2 be non-adjacent vertices of graph G. Form a graph H from G by replacing vertices u_1 and u_2 with a single vertex u_{12}. Any edge of G joining two vertices different from u_1 and u_2 is an edge of H. Also if wu_1 or wu_2 is an edge of G, for any vertex w different from u_1 and u_2, then wu_{12} is an edge of H. We say H is formed from G by ***identifying*** u_1 and u_2.

The ***ceiling*** of any real number x, denoted by $\lceil x \rceil$, is the smallest integer greater than or equal to x, and the ***floor*** of x, denoted by $\lfloor x \rfloor$, is the largest integer less than or equal to x. For example $\lceil 4.32 \rceil = 5$, and $\lfloor 4.32 \rfloor = 4$.

Let $\mathbb{N}$ be the set of all positive integers. A ***proper coloring*** of the vertices of a graph G is a function $f : V(G) \to \mathbb{N}$ such that if u and v are adjacent then $f(u) \neq f(v)$. We call f(u) the color of u. The ***chromatic number***, $\chi(G)$, is the smallest number of colors that can be used in a proper coloring of G. The ***chromatic sum***, $\sum(G)$, is a recent variation introduced in the dissertation of Ewa Kubicka [5]. It is the smallest possible total over all vertices, $\sum_{v \in V(G)} f(v)$, that can occur among all proper colorings of G using natural numbers. For any coloring f, the sum $\sum_{v \in V(G)} f(v)$ is also called the ***cost*** of coloring f. A proper coloring c of a graph G is called a ***best coloring*** of G whenever

$\sum_{v \in V} c(v) = \sum(G)$. The ***strength*** s(G) of a graph G is the minimum number of colors needed to obtain a best coloring.

Preview

In this thesis, we study some aspects of the fairly new concept of chromatic sums. As we have already seen, the minimum sum of colors might be achieved by using more than the minimum number of colors. In Chapter Two we first describe the construction of a family of trees which on one hand have arbitrarily large strengths and on the other hand are of the smallest order. Then we go on to calculate the order and maximum degree of these trees and compare these parameters for various constructions.

Even though the trees constructed in Chapter Two are the smallest possible, they have a huge maximum degree. In Chapter Three we study a construction of trees with arbitrarily large chromatic sums, large orders, but comparatively smaller maximum degrees.

Lastly, in Chapter Four we describe the construction of two classes of graphs that are not trees but still have arbitrarily large chromatic sums. At the end we compare the strengths and orders of the two constructions.

Chapter Two

Kubicka's Construction

Having introduced the idea of chromatic sum, it is interesting to note that the minimum sum of colors might very well be achieved by using more than the minimum number of colors, i.e. s(G) can be equal to or greater than $\chi(G)$. One might think that a minimum cost coloring can be obtained by selecting a proper coloring with the minimum number of colors and then giving the largest color class color 1, the next color 2, and so on. However, even among trees, which are bipartite and hence have chromatic number 2, more colors may be needed to obtain a minimum cost coloring. On page 2 we have shown such an example where $s(T) = 3 > \chi(T) = 2$. The example shows that sometimes we are forced to use additional colors to obtain the chromatic sum. In fact s(G) may be arbitrarily large even when $\chi(G) = 2$.

In this chapter we will construct a family of trees T_k^m to demonstrate that for each k, some trees need k colors to achieve their chromatic sum. In fact, we shall prove that in our family of trees we have the smallest tree in which color k is forced to appear in every best coloring. This work appeared in Kubicka and Schwenk's paper [4].

Construction of the tree T_k^m

We recursively construct rooted tree T_k^m and then show that T_k^m has strength k, that in any best coloring of T_k^m color k must be used on the root, and that any change from color k on the root to a lower color will increase the total cost by at least m.

Let T_1^1 be the rooted tree with one vertex. We now construct T_k^m, $k \geq 2$, recursively by joining a root r to various copies of T_i^1. Specifically T_k^m is the unique tree such that $T_k^m - r = \bigcup_{i=1}^{k-1} (m + k - i)T_i^1$. Some examples are shown in Figure 2.1 below.

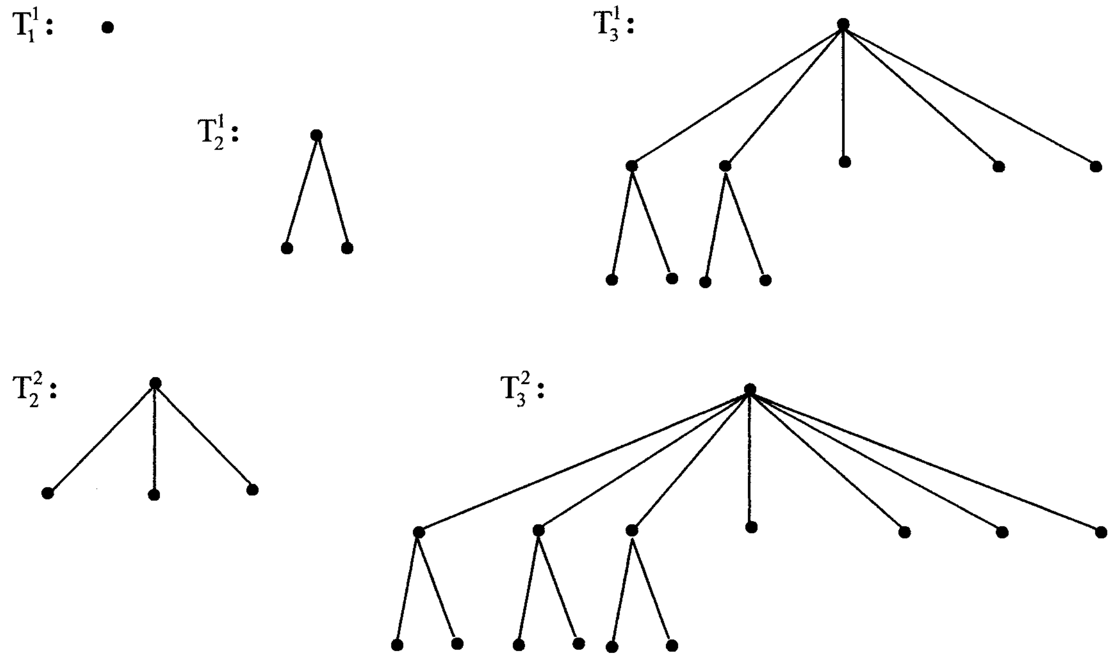

Figure 2.1. Examples of T_k^m.

In general, T_k^m is:

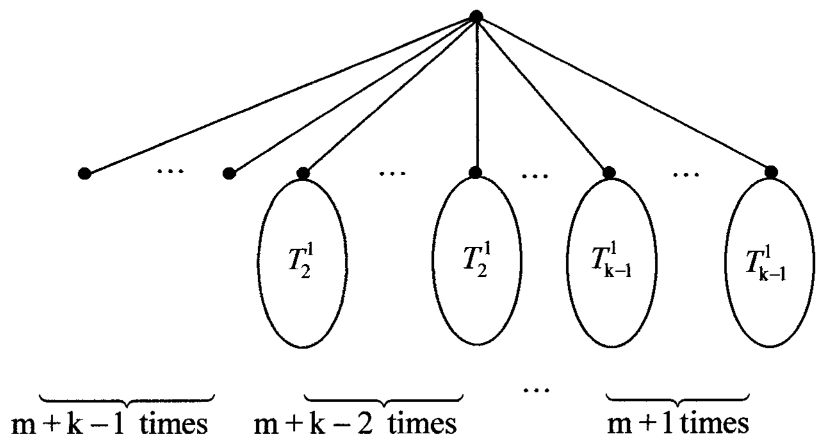

Figure 2.2. General graph of T_k^m.

Theorem 2.1. *For $k \geq 2$, the tree T_k^m is the unique smallest rooted tree for which the following hold:*

a) The strength of T_k^m is k.

b) In every best coloring, color k is forced to appear at the root, and

c) Any change of the color k at the root to a lower color must increase the total cost by at least m.

Proof. (by induction on k)

Base Condition:

$$\text{Note that } T_2^m - r = \bigcup_{i=1}^{1} (m+2-i)T_i^1$$

$$= (m+1)\,T_1^1.$$

Hence T_2^m looks like the following:

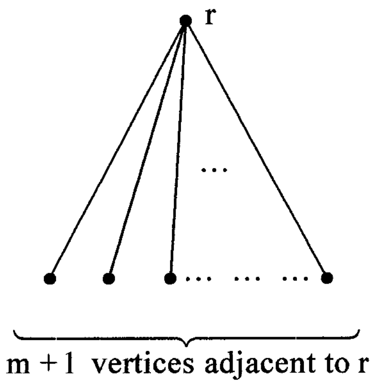

Figure 2.3. Graph of T_2^m.

It is clear that the strength of T_2^m is 2. Also, it is easy to see that T_2^m has to have color 2 at the root in any best coloring, and changing it to 1 increases the total cost by m. To complete the base case, we have to show that T_2^m is the smallest such tree. Note that $|T_2^m| = m + 2$. Let T be a smallest rooted tree of strength 2, with the root colored 2, such that reducing the color on the root increases the total cost by at least m. Also let c be the corresponding best coloring of tree T, and b be the number of vertices colored 2 by c.

Lemma 1. Tree T has at least b + m vertices colored 1 by c.

Proof. On the contrary, we assume that T has $b + m - s$, $s > 0$, vertices colored 1. Hence the total cost is $2b + (b + m - s) = 3b + m - s$. After interchanging colors, the total cost is $2(b + m - s) + 1b = 3b + 2m - 2s$. The difference in cost is $(3b + 2m - 2s) - (3b + m - s) = m - s < m$, which contradicts the fact that changing the color on the root costs at least m. Thus, T has at least b + m vertices colored 1, which proves the lemma.

By Lemma 1 we obtain $|T| \geq 2b + m \geq m + 2$. If $2b + m > m + 2$, we have a contradiction to the fact that T is a smallest rooted tree, therefore $|T| = 2b + m = m + 2$, and it follows that $T = T_2^m$.

Induction Hypothesis:

Assume that the theorem is true for T_j^m for each j, $2 \leq j \leq k$. Let T(i, m) denote a tree of smallest order of strength i, where color i must be used on the root and the use of any smaller color on the root increases the total cost by at least m.

Induction Step:

Consider T(k + 1, m) and a best coloring c of it. We will show that T(k + 1, m) must be T_{k+1}^m. After removing the root from T(k + 1, m), we are left with a forest of rooted trees. Let $F_1(k + 1, m)$ denote the subforest containing all those trees with roots colored 1.

Lemma 2. $F_1(k + 1, m)$ is a smallest forest with the property that changing the color 1 at the roots to any other color increases the cost by at least k + m.

Proof. First we assume, on the contrary, that changing the color 1 at the roots of $F_1(k + 1, m)$ to any other color costs t which is less than k + m (i.e. $t < k + m$). Let r be the root of T(k + 1, m). Now, change the color at r from (k + 1) to 1, thus saving a cost k. Since the change of color on $F_1(k + 1, m)$ costs $t < k + m$, the total change in cost of coloring T(k + 1, m) is $t - k < m$ which is a contradiction to the definition of tree T(k + 1, m).

Now assume, on the contrary, that there exists a smaller subforest with the property that changing the color 1 at the roots to any other color costs at least $k + m$. Then there exists a smaller graph compared to the original graph $T(k + 1, m)$. This is a contradiction because $T(k + 1, m)$ is the smallest tree of its kind. Hence the lemma is proved.

We also notice that $F_1(k, m + 1)$, a subforest of $T(k, m + 1)$, has the same properties as $F_1(k + 1, m)$. That is, changing color 1 at the roots to any other color costs at least $k + m$. The proof is similar to the above. Assume, on the contrary, that changing the color 1 at the roots of $F_1(k, m + 1)$ to any other color costs p which is less than $k + m$ (i.e. $p < k + m$). Now, change the color at the root of $T(k, m + 1)$ from k to 1, thus saving a cost $(k - 1)$. Since the change of color on $F_1(k, m + 1)$ costs $p < k + m$, the total change in cost of $T(k, m + 1)$ is $p - (k - 1) < m + 1$ which is a contradiction to the definition of tree $T(k, m + 1)$. Also, $F_1(k, m + 1)$ is a smallest forest such that changing color 1 at the roots to any other color costs at least $k + m$. Now $F_1(k, m + 1)$ and $F_1(k + 1, m)$ each have the property that it is a smallest subforest such that changing the color 1 at the roots increases the cost by at least $k + m$. By the induction hypothesis we know that

$T(k, m + 1) = T_k^{m+1}$, and because $T_k^{m+1} = \bigcup_{i=1}^{k-1}(m+1+k-i)T_i^1$, we have

$F_1(k, m + 1) = (k + m)\ T_1^1$. That is $F_1(k, m + 1)$ is simply a forest of $k + m$ isolated vertices. Since $F_1(k + 1, m)$ has the same properties as $F_1(k, m + 1)$, it follows that

$F_1(k + 1, m) = F_1(k, m + 1)$.

Let $F_i(k+1, m)$, $1 \le i \le k$, be the subforest of $T(k+1, m)$ containing all trees with roots colored i. We have already shown that $F_1(k+1, m) = (k+m)\ T_1^1$. If we show $F_i(k+1, m) = (k+1+m-i)\,T_i^1$ for $1 \le i \le k$, then clearly $T(k+1, m) = T_{k+1}^m$, which will complete the proof of the theorem.

Specifically we show $F_k(k+1, m) = (m+1)\ T_k^1$, the argument for other $F_i(k+1, m)$ being analogous. Now similar to Lemma 2, $F_k(k+1, m)$ is a smallest forest with the property that changing the color k at the roots to other colors costs at least m + 1. Also by the induction hypothesis the forest $(m+1)\,T_k^1$ has all the roots colored k, and any change of that color costs at least m + 1. Therefore $|\,F_k(k+1, m)| \le (m+1)\,|\,T_k^1|$. Now we will prove that $|\,F_k(k+1, m)| \ge (m+1)\,|\,T_k^1|$.

Consider the subtree T_0 of $T(k+1, m)$ consisting of the largest connected component of $T(k+1, m)$ containing the root and only vertices colored k + 1 and k. Among all possible $T(k+1, m)$, we select one which has the fewest number of vertices colored k occurring in the corresponding subtree T_0. Let b denote the number of vertices of T_0 colored k + 1.

Lemma 3. T_0 must have at least m + b vertices colored k.

Proof. On the contrary, let the number of vertices colored k in T_0 be $t < m + b$. Now, if we change vertices colored k + 1 to k then we save b, and changing vertices colored k to k + 1 costs $t < m + b$. Therefore the net cost change for $T(k+1, m)$ is $t - b < m + b - b = m$, a contradiction to the properties of

T(k + 1, m). Hence, the lemma is proved.

Now, call the vertices colored k in T_0 as $v_1, v_2, \ldots, v_r$, $(r \geq m+b)$. Let S_i denote the subtree of T(k + 1, m) with root v_i which is formed after deleting all edges between v_i and any vertex colored k + 1.

Lemma 4. There must be at least m + 1 of these subtrees S_i's for which $|S_i| \geq |T_k^1|$.

Proof. Assume not, i.e. let there be fewer than m + 1 S_i's for which $|S_i| \geq |T_k^1|$. Recall that T_k^1 is the smallest rooted tree with strength k in which color k is forced to appear at the root and any change of this color k to a lower color must increase the total cost by at least 1.

Thus $|S_i| < |T_k^1|$ means that S_i has a best coloring c' which uses a different color at the root, either smaller than k or larger than k.

<u>Case 1</u>. At least one of the S_i's, where $|S_i| < |T_k^1|$, has a best coloring c' which uses a color smaller than k on its root. In this case we can change our best coloring of T(k + 1, m) by just using c' on this S_i obtaining a best coloring of T(k + 1, m) with smaller number of vertices colored k in T_0, a contradiction to our definition of T_0.

<u>Case 2</u>. The color on the root of one of these S_i's, where $|S_i| < |T_k^1|$, is at least k + 2 in a best coloring c'. Then the strength of T(k + 1, m) is at least k + 2, a contradiction.

Case 3. Each S_i with $|S_i| < |T_k^1|$ has a best coloring c' with root colored $k + 1$. Thus the color on the roots of these S_i's can be changed from k to k + 1 at no cost. So we swap colors k + 1 and k throughout T_0. This produces no change in cost on these smaller S_i's, whereas it produces a total change of at most m on the other S_i's, and reduces the cost at each vertex colored k + 1 by 1. Thus the change in cost is less than m, a contradiction to our definition of T(k + 1, m). Thus the lemma is proved.

Therefore $|F_k(k + 1, m)| \geq (m + 1)|T_k^1|$. Hence we can now conclude that $|F_k(k + 1, m)| = (m + 1)|T_k^1|$. Furthermore we now know that T(k + 1, m) has exactly (m + 1) copies of S_i and for each i, $|S_i| = |T_k^1|$ and the root of each S_i is colored k.

Lemma 5. $S_i = T_k^1 \ \forall$ i.

Proof. Assume otherwise. Recolor the root of T(k + 1, m) with k and recolor all roots of the S_i's with a color smaller than k. For each S_i which is a T_k^1, the cost of recoloring is at least 1. For each S_i which is not a T_k^1, the cost is at least 2 because T_k^1 is unique. Therefore the total cost of the recoloring is at least m + 1, contradicting the definition of T(k + 1, m). Thus Lemma 5 is proved.

Therefore $F_k(k + 1, m) = (m + 1)\ T_k^1$ which implies $T(k + 1, m) = T_{k+1}^m$, and this completes the proof of Theorem 2.1.□

Denote by T_k the unrooted tree formed by adding an edge between the roots of two copies of T^2_{k-1}.

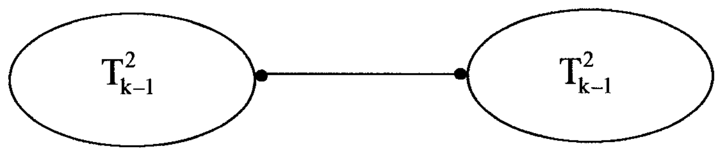

Figure 2.4. Graph of T_k.

We let t^m_k denote $|T^m_k|$.

Theorem 2.2. *T_k is the unique smallest tree in which color k is needed in every best coloring.*

Proof. Let F_k be a smallest tree that requires k colors in every best coloring, and suppose we have a best coloring of F_k. Let $e = v_1v_2$ be an edge joining v_1 (colored k) with vertex v_2 (colored $k - 1$). Such an edge exists, since if not, then v_1 can be recolored $k - 1$, a contradiction to the fact that we started with a best coloring of F_k. Removing edge e leaves two trees S_1 and S_2 which contain vertices v_1 and v_2 respectively. Now in order to prove that T_k is the smallest tree in which color k is needed in every best coloring (i.e. $T_k = F_k$), the following lemmas are proved.

Lemma 1. There is a best coloring of F_k, with exactly one vertex, say v_1, colored k.

Proof. Suppose, on the contrary, that there exist two such vertices u_1 and v_1 colored k. Remove u_1 from the graph F_k and let S be the component containing

v_1. Then S is smaller than F_k which is a smallest tree requiring (at least) k colors in every best coloring. Therefore there exists a best coloring of S that uses less than k colors. Keeping this best coloring on S and the original coloring on $F_k - S$, we now have a best coloring of F_k which is either less costly than the original coloring, a contradiction, or has the same cost as the original best coloring, but with one less vertex colored k. If there are more than two vertices colored k, then by continuing this process we can find a best coloring of F_k using k colors which has only one vertex colored k. Hence Lemma 1 is proved.

Lemma 2. Each best coloring of F_k with only one vertex v_1 colored k, has only one adjacency of v_1 colored $k - 1$.

Proof. Suppose, on the contrary, that u_1 and u_2 are adjacent to v_1 and have color $k - 1$ in some best coloring c with v_1 the only vertex colored k. Now form a new tree F from F_k by identifying u_1 and u_2, and call this new vertex u_{12}. Tree F is smaller than F_k by one vertex and thus has a best coloring c' which requires less than k colors. Since c' is a best coloring, the total cost of it on F is less than or equal to the cost of applying c to F. Now split u_{12} back to u_1 and u_2 to reform F_k keeping the coloring c' on F_k. The resulting coloring uses less than k colors.

Let C(G) be the total cost of the coloring c(G), where G is any arbitrary graph.

Then $C'(F) \le C(F)$. Therefore

$$C'(F_k) = C'(F) + c'(u_{12}) \le C(F) + c'(u_{12}) \le C(F) + k - 1 = C(F_k).$$

This contradicts the fact that every best coloring of F_k uses k colors, thus Lemma 2 is proved.

Lemma 3. Given any best coloring c of F_k with only one vertex v_1 colored k and only one vertex v_2 colored $k-1$, we still have a best coloring of S_2 when edge $e = v_1v_2$ is removed.

Proof. Assume we do not have a best coloring of S_2. Then the following cases arise:

<u>Case 1</u>. If a best coloring of S_2 uses fewer than $k-1$ colors, then recolor v_1 with $k-1$. Thus the cost of coloring F_k has been reduced, a contradiction.

<u>Case 2</u>. If a best coloring c' of S_2 uses more than $k-1$ colors, then since c applied to S_2 is not a best coloring, it follows that c' has smaller cost than c applied to S_2. Applying c' to S_2 and c to S_1 reduces the cost of coloring F_k, a contradiction.

If neither case 1 nor case 2 occurs, then every best coloring of S_2 uses k colors, which contradicts the minimality of F_k. Thus Lemma 3 is proved.

Lemma 4. Any best coloring of S_1 has v_1 colored $k-1$.

Proof. Assume otherwise. Then there exists a best coloring of S_1 with v_1 colored more than $k-1$ or v_1 colored less than $k-1$. In the latter case we have a contradiction since we can use this best coloring of S_1 together with c applied to S_2 to give a best coloring of F_k with less than k colors.

In $c(S_1)$ there is no vertex colored $k - 1$ adjacent to v_1. Therefore we can use $k - 1$ on v_1 and reduce the cost of coloring S_1. Thus if there exists a best coloring of S_1 using a color greater than $k - 1$ on v_1, then this best coloring is as cheap as that given in the previous sentence. Therefore use it together with c on S_2 to reduce the cost of F_k, a contradiction. Hence Lemma 4 is proved.

Using Lemmas 3 and 4 and starting with c applied to S_2 and any best coloring of S_1, the following lemmas are proved.

Lemma 5. If we change the color of v_1 to a smaller color, the cost increase is at least 2 on S_1.

Proof. On the contrary, if the increased cost is one or less, then by Lemma 4 the cost increase has to be 1. Now rejoin v_1 and v_2 with edge e. Using coloring c on S_2 and this new coloring of S_1, F_k now has a coloring with less than k colors at a cost no more than that of a best coloring, a contradiction.

Lemma 6. If we change v_2 to a smaller color than that given by c, then the cost increase is at least 2 on S_2.

Proof. Note that there is a cost increase, for otherwise we can use the new coloring on S_2, $k - 1$ on v_1, and c on $S_1 - v_1$. This results in a smaller cost coloring of F_k, a contradiction. If the cost increase is only 1, then again using $k - 1$ on v_1 and c on $S_1 - v_1$ we have a best coloring of F_k with $k - 1$ colors, a contradiction, which proves Lemma 6.

Hence from Lemmas 5 and 6, S_1 and S_2 have the properties of T^2_{k-1}. Also, since F_k has minimum order, S_1 and S_2 are the same as T^2_{k-1}. Hence $F_k = T_k$, which proves Theorem 2.2.□

Order of Kubicka's Graphs

Theorem 2.3. *For $k \geq 2$, the order of T_k is given by*

$$2\left|T^2_{k-1}\right| = 2t^2_{k-1} = \frac{1}{\sqrt{2}}[(2+\sqrt{2})^{k-1} - (2-\sqrt{2})^{k-1}].$$

Proof. The equation defining these trees immediately gives the recurrence

$$t^2_k = 1 + \sum_{i=1}^{k-1}(k+2-i)t^1_i \qquad (2.1)$$

We show, by induction on k, that the recurrence relation $t^2_k = 4t^2_{k-1} - 2t^2_{k-2}$ holds.

Base Condition:

We know from equation 2.1 above that $t^2_2 = 4$, $t^2_3 = 14$, and $t^2_4 = 48$, and we notice that $4t^2_3 - 2t^2_2 = 48 = t^2_4$. Hence the recurrence relation holds for $k = 4$.

Induction Hypothesis:

Assume that $k \geq 5$, and that the recurrence relation has already been proved for all i, $4 \leq i < k$.

Induction Step:

Note that

$$t_k^2 - 4t_{k-1}^2 + 2t_{k-2}^2 = 1 + \sum_{i=1}^{k-1}(k+2-i)t_i^1 - 4 - 4\sum_{i=1}^{k-2}(k+1-i)t_i^1 + 2 + 2\sum_{i=1}^{k-3}(k-i)t_i^1 .$$

Adjusting the indices in the second and third summations and collecting the isolated terms, we get

$$t_k^2 - 4t_{k-1}^2 + 2t_{k-2}^2 = -1 + \sum_{i=1}^{k-1}(k+2-i)t_i^1 - 4\sum_{i=2}^{k-1}(k+2-i)t_{i-1}^1 + 2\sum_{i=3}^{k-1}(k+2-i)t_{i-2}^1$$

$$= -1 + (k+1)t_1^1 + k\,t_2^1 + (k-1)\,t_3^1 - 4k\,t_1^1 - 4(k-1)\,t_2^1 + 2(k-1)\,t_1^1$$

$$+ \sum_{i=4}^{k-1}(k+2-i)t_i^1 - 4\sum_{i=4}^{k-1}(k+2-i)t_{i-1}^1 + 2\sum_{i=4}^{k-1}(k+2-i)t_{i-2}^1$$

$$= 0 + \sum_{i=4}^{k-1}(k+2-i)(t_i^1 - 4t_{i-1}^1 + 2t_{i-2}^1)\text{, because } t_1^1 = 1,\ t_2^1 = 3,\ t_3^1 = 10 .$$

The final summation equals zero since the induction hypothesis guarantees that all the terms of the form $t_i^1 - 4t_{i-1}^1 + 2t_{i-2}^1$ have value 0 for all i, $4 \le i < k$. Hence, $t_k^2 = 4t_{k-1}^2 - 2t_{k-2}^2$.

Now, it's left to solve the recurrence relation $t_k^2 = 4t_{k-1}^2 - 2t_{k-2}^2$ with the initial conditions $t_2^2 = 4$, and $t_3^2 = 14$, $k \ge 2$. The characteristic equation we obtain is $x^{k-2}(x^2 - 4x + 2) = 0$, i.e. $x^2 - 4x + 2 = 0$, the roots of which are $\lambda_1 = 2 + \sqrt{2}$, and $\lambda_2 = 2 - \sqrt{2}$. Therefore the recurrence relation $t_k^2 = 4t_{k-1}^2 - 2t_{k-2}^2$ has the general solution $t_k^2 = (2+\sqrt{2})^k A_1 + (2-\sqrt{2})^k A_2$. With the initial conditions we get the following system of linear equations:

$$t_2^2 = 4 = (2+\sqrt{2})^2 A_1 + (2-\sqrt{2})^2 A_2$$

$$t_3^2 = 14 = (2+\sqrt{2})^3 A_1 + (2-\sqrt{2})^3 A_2.$$

The solution to the above is given by $A_1 = \frac{1}{2\sqrt{2}}$, and $A_2 = -\frac{1}{2\sqrt{2}}$. Hence the exact solution to the recurrence relation with the given initial condition is

$$|T_k^2| = t_k^2 = \frac{1}{2\sqrt{2}}[(2+\sqrt{2})^k - (2-\sqrt{2})^k].$$

Of course, by definition of T_k, $|T_k| = 2t_{k-1}^2 = \frac{1}{\sqrt{2}}[(2+\sqrt{2})^{k-1} - (2-\sqrt{2})^{k-1}]$. Hence Theorem 2.3 is proved.□

Theorem 2.4. *Tree T_k^m, $k \geq 2$ has maximum degree $m(k-1) + \frac{k(k-1)}{2}$.*

Proof. The root of T_k^m clearly is the vertex of maximum degree.

Since, $T_k^m - r = \bigcup_{i=1}^{k-1} (m+k-i)\, T_i^1$, $d_{T_k^m}(r) = \sum_{i=1}^{k-1} (m+k-i)$

$$= (m+k-1) + (m+k-2) + \ldots + (m+1),\ m \geq 1$$

$$= m(k-1) + \frac{k(k-1)}{2}.$$ □

Corollary 2.5. Tree T_k, $k \geq 2$, has maximum degree approximately $\frac{k^2}{2}$.

For the various values of m, the smallest T_k^m is clearly T_k^1. We first find a simple upper bound for the order of T_k^1 and then find a somewhat more complicated exact value for the number of vertices in T_k^1. Let t_i denote $|T_i^1|$.

Theorem 2.6. *For all integral values of k,* $t_k \leq \left(\frac{7}{2}\right)^k$.

Proof. It can be easily verified that $t_1 = 1$, $t_2 = 3$ and $t_3 = 10$. From the definition of T_k^1 we know $t_k = 1 + \sum_{i=1}^{k-1}(1+k-i)t_i$

$$= 1 + k t_1 + (k-1)\, t_2 + (k-2)\, t_3 + (k-3)\, t_4 + \ldots + 3 t_{k-2} + 2 t_{k-1}$$

$$= (1 + (k-1)\, t_1 + (k-2)\, t_2 + (k-3)\, t_3 + \ldots + 2 t_{k-2})$$

$$+ (t_1 + t_2 + t_3 + \ldots + t_{k-2}) + 2 t_{k-1}$$

$$= t_{k-1} + (t_1 + t_2 + t_3 + \ldots + t_{k-2}) + 2 t_{k-1}$$

$$= (t_1 + t_2 + t_3 + \ldots + t_{k-2}) + 3 t_{k-1}.$$

Since $k - i \geq 2$ for $0 \leq i \leq k-2$ and $k \geq 2$,

$$t_{k-1} = 1 + (k-1)\, t_1 + (k-2)\, t_2 + (k-3)\, t_3 + \ldots + 2 t_{k-2} \geq 2(t_1 + t_2 + t_3 + \ldots + t_{k-2}).$$

This yields,

$$t_k < \frac{1}{2} t_{k-1} + 3 t_{k-1} = \frac{7}{2}\, t_{k-1} \leq \frac{7}{2}\left(\frac{7}{2}\, t_{k-2}\right) \leq \ldots \leq \left(\frac{7}{2}\right)^{k-1} t_{k-(k-1)} = \left(\frac{7}{2}\right)^{k-1} t_1 = \left(\frac{7}{2}\right)^{k-1}.$$

Hence an upper bound for t_k is $\left(\frac{7}{2}\right)^{k-1}$. □

Before calculating an exact value of t_k, we look at some initial t_i's:

$t_1 = 1;$

$t_2 = 1 + 2t_1 = 3;$

$t_3 = 1 + 3t_1 + 2t_2 = 1 + 3 + 6 = 10.$

We will look at the next few t_i's in terms of t_1, t_2, and t_3.

$t_4 = 1 + 4t_1 + 3t_2 + 2t_3 = 1 + 4 + 9 + 20 = 34;$

$t_5 = 1 + 5t_1 + 4t_2 + 3\,t_3 + 2t_4$

$\quad = 3 + 13t_1 + 10t_2 + 7t_3 = 3 + 13 + 30 + 70 = 116;$

$t_6 = 1 + 6t_1 + 5t_2 + 4t_3 + 3t_4 + 2t_5$

$\quad = 10 + 44t_1 + 34t_2 + 24t_3 = 10 + 44 + 102 + 240 = 396.$

We notice that $t_k = t_{k-3} + (t_{k-2} + t_{k-3})t_1 + t_{k-2}t_2 + (t_{k-2} - t_{k-3})t_3 = 14t_{k-2} - 8t_{k-3}$ for $k = 4$, 5, and 6, which leads us to the following theorem.

Theorem 2.7. *$t_k = 14t_{k-2} - 8t_{k-3}$, for $k \geq 4$.*

Proof. In order to prove the theorem, we simply show that the right-hand side (RHS) and left-hand side (LHS) are equal.

RHS

$= 14t_{k-2} - 8t_{k-3}$

$= 14(1 + \sum_{i=1}^{k-3}(k-i-1)t_i) - 8t_{k-3}$

$= [14(1 + (k-2)t_1 + (k-3)t_2 + \ldots + (k-i-1)t_i + \ldots + 2t_{k-3})] - 8t_{k-3}$

$$= 14(1+(k-2)t_1+(k-3)t_2+\ldots+(k-i-1)t_i+\ldots+3t_{k-4})+28t_{k-3}-8t_{k-3}$$

$$= 14(1+\sum_{i=1}^{k-4}(k-i-1)t_i)+20(1+\sum_{i=1}^{k-4}(k-i-2)t_i)$$

$$= 14+20+\sum_{i=1}^{k-4}[14(k-i-1)+20(k-i-2)]t_i$$

$$= 34+\sum_{i=1}^{k-4}(34k-34i-54)t_i.$$

LHS

$$= t_k$$

$$= 1+\sum_{i=1}^{k-1}(k-i+1)t_i$$

$$= (1+\sum_{i=1}^{k-4}(k-i+1)t_i)+4t_{k-3}+3t_{k-2}+2t_{k-1}$$

$$= (1+\sum_{i=1}^{k-4}(k-i+1)t_i)+4(1+\sum_{i=1}^{k-4}(k-i-2)t_i)+3(1+\sum_{i=1}^{k-4}(k-i-1)t_i+2t_{k-3})$$

$$+2(1+\sum_{i=1}^{k-4}(k-i)t_i+3t_{k-3}+2t_{k-2})$$

$$= (1+\sum_{i=1}^{k-4}(k-i+1)t_i)+4(1+\sum_{i=1}^{k-4}(k-i-2)t_i)+3(1+\sum_{i=1}^{k-4}(k-i-1)t_i)$$

$$+2(1+\sum_{i=1}^{k-4}(k-i)\ t_i)+12t_{k-3}+4t_{k-2}$$

$$= (1+\sum_{i=1}^{k-4}(k-i+1)t_i)+4(1+\sum_{i=1}^{k-4}(k-i-2)t_i)+3(1+\sum_{i=1}^{k-4}(k-i-1)t_i)+2(1+\sum_{i=1}^{k-4}(k-i)t_i)$$

$$+12t_{k-3}+4(1+\sum_{i=1}^{k-4}(k-i-1)t_i+2t_{k-3})$$

$$= (1+\sum_{i=1}^{k-4}(k-i+1)t_i)+4(1+\sum_{i=1}^{k-4}(k-i-2)t_i)+3(1+\sum_{i=1}^{k-4}(k-i-1)t_i)+2(1+\sum_{i=1}^{k-4}(k-i)t_i)$$

$$+20t_{k-3}+4(1+\sum_{i=1}^{k-4}(k-i-1)t_i)$$

$$= (1+\sum_{i=1}^{k-4}(k-i+1)t_i)+4(1+\sum_{i=1}^{k-4}(k-i-2)t_i)+3(1+\sum_{i=1}^{k-4}(k-i-1)t_i)+2(1+\sum_{i=1}^{k-4}(k-i)t_i)$$

$$+20(1+\sum_{i=1}^{k-4}(k-i-2)t_i)+4(1+\sum_{i=1}^{k-4}(k-i-1)t_i)$$

$$=(1+4+3+2+20+4)$$

$$+\sum_{i=1}^{k-4}[(k-i+1)+4(k-i-2)+3(k-i-1)+2(k-i)+20(k-i-2)+4(k-i-1)]\ t_i$$

$$=34+\sum_{i=1}^{k-4}(34k-34i-54)t_i$$

= RHS.

Hence the theorem is proved.□

Theorem 2.8. *For each $k \geq 1$, $t_k = \left\lceil .25\left(2+\sqrt{2}\right)^k \right\rceil$.*

Proof. We solve the recurrence relation $t_k = 14t_{k-2} - 8t_{k-3}$ to find an exact value of t_k. The characteristic equation we obtain is $x^{k-3}(x^3-14x+8) = 0$, i.e. $x^3-14x+8=0$, which can be factorized into $(x+4)(x^2-4x+2) = 0$. The roots of the characteristic equation are $\lambda_1=-4$, $\lambda_2=2+\sqrt{2}$, and $\lambda_3=2-\sqrt{2}$. Therefore the recurrence relation has the general solution $t_k=-4^kA_1+(2+\sqrt{2})^kA_2+(2-\sqrt{2})^kA_3$. With the initial conditions $t_1=1$, $t_2=3$, and $t_3=10$, we obtain the following system of linear equations:

$$t_1 = 1 = -4A_1 + (2+\sqrt{2})A_2 + (2-\sqrt{2})A_3$$

$$t_2 = 3 = -4^2 A_1 + (2+\sqrt{2})^2 A_2 + (2-\sqrt{2})^2 A_3$$

$$t_3 = 10 = -4^3 A_1 + (2+\sqrt{2})^3 A_2 + (2-\sqrt{2})^3 A_3.$$

The solution to the above is given by $A_1 = 0,\ A_2 = A_3 = 0.25$. Hence the exact solution to the recurrence relation with the given initial condition is $t_k = 0.25[(2+\sqrt{2})^k + (2-\sqrt{2})^k]$. Since $0 < (2-\sqrt{2})^k < 1$ and $k \geq 1$, $t_k = \lceil 0.25(2+\sqrt{2})^k \rceil$, and thus the theorem is proved.□

Chapter Three

The Construction of Jiang and West

In Kubicka's construction in Chapter Two, the tree with strength k has maximum degree about $\frac{k^2}{2}$. In the following theorem, by Jiang and West [4], we construct for each $k \geq 1$ a tree T_k with strength k and maximum degree $2k - 2$. Given a proper coloring f of a tree T, we use $\sum f$ to denote $\sum_{v \in V(T)} f(v)$.

Construction

Linearly order the pairs of natural numbers so that $(p, q) < (i, j)$ if $p + q < i + j$, or if $p + q = i + j$ and $q < j$. With respect to this ordering, we inductively construct, for each pair $(i, j) \in \mathbb{N} \times \mathbb{N}$, a rooted tree T_i^j and a coloring f_i^j of T_i^j. In other words, we construct trees in the order $T_1^1, T_2^1, T_1^2, T_3^1, T_2^2, T_1^3, \ldots$. Our desired tree with strength k will be T_k^1. Let $[n] = \{k \in \mathbb{Z} : 1 \leq k \leq n\}$.

Let T_1^1 be a tree of order 1, and let f_1^1 assign color 1 to this single vertex. Consider $(i, j) \neq (1, 1)$, and suppose that for each $(p, q) < (i, j)$ we have constructed tree T_p^q and coloring f_p^q. We construct T_i^j and f_i^j inductively as follows. Let u be the root of T_i^j. For each k such that $1 \leq k \leq i + j - 1$ and $k \neq i$, we take two copies of T_k^m, where

$m = \lceil (i + j - k) / 2 \rceil$. Note that we will have $(i + j - 1) - 1 = i + j - 2$ different values of k. We join each root of these trees to u to form T_i^j (see Figure 3.2). We define the coloring f_i^j of T_i^j by assigning i to the root u and using f_k^m on each copy of T_k^m rooted at a child of u. See Figure 3.1 for examples of T_i^j's.

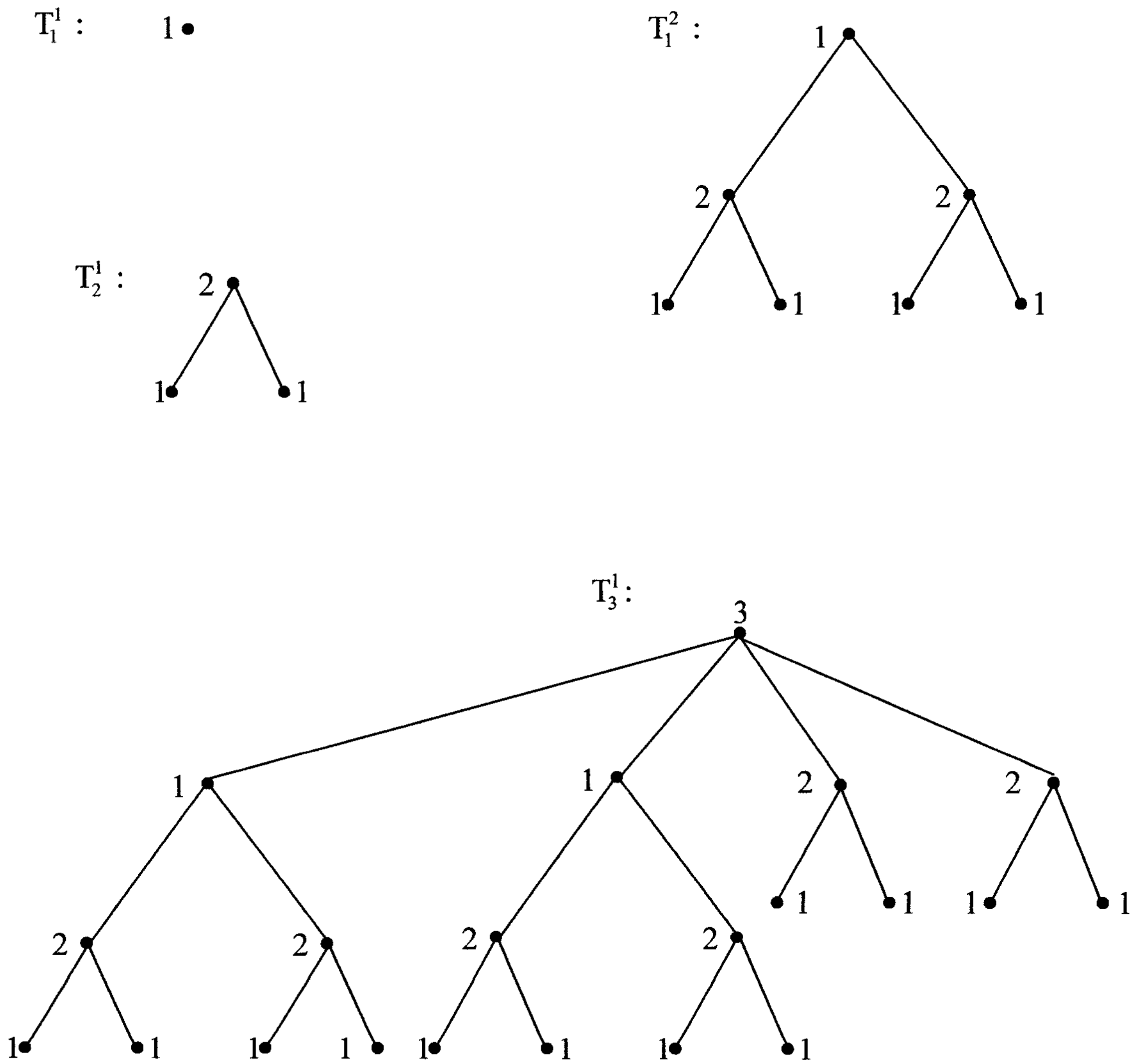

Figure 3.1. Specific examples of T_i^j.

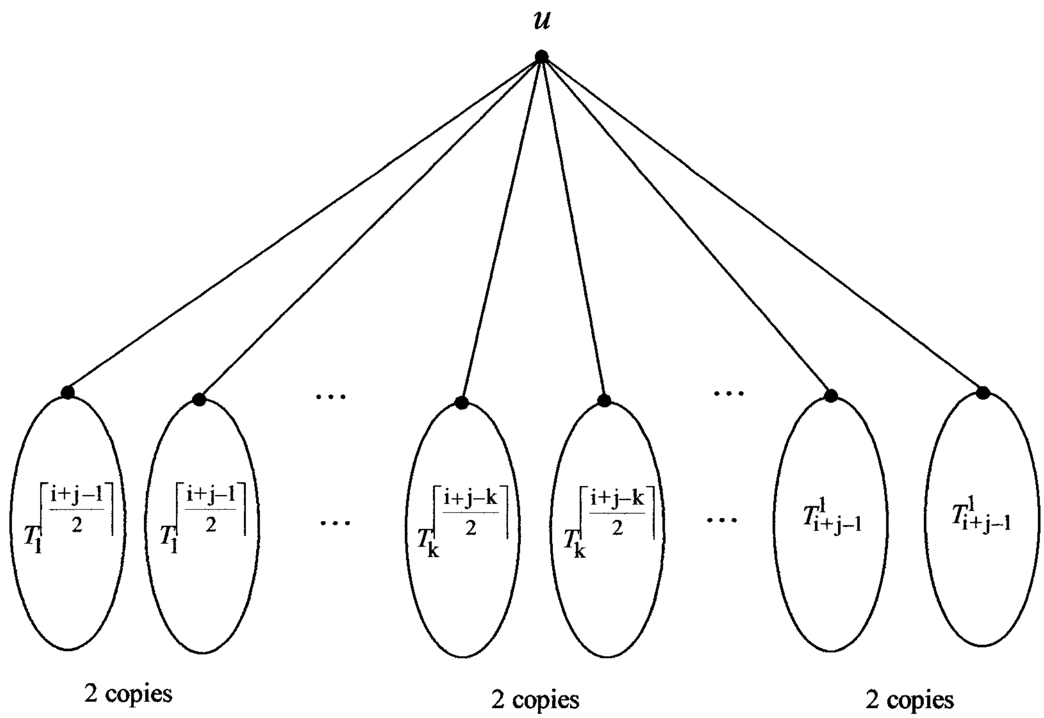

Figure 3.2. Construction of T_i^j.

Lemma 3.1. *For (i, j) $\in \mathbb{N} \times \mathbb{N}$ the construction of T_i^j is well defined, and f_i^j is a proper coloring of T_i^j with color i at the root.*

Proof. To show that T_i^j is well defined, it suffices to show that when $(i, j) \neq (1, 1)$, every tree used in the construction of T_i^j has been constructed previously. We use trees of the form T_k^m, where $k \in [i + j - 1] - \{i\}$ and $m = \lceil (i + j - k) / 2 \rceil$. It suffices to show that $k + m \le i + j$ and that $m < j$ when $k + m = i + j$.

For the first statement, we have

$$k + m = k + \lceil (i + j - k) / 2 \rceil = \lceil (2k + (i + j - k)) / 2 \rceil$$

$$= \lceil (i + j + k) / 2 \rceil$$

$$\leq \lceil (i + j + (i + j - 1)) / 2 \rceil, \text{ since } 0 < k \leq i + j - 1$$

$$= i + j.$$

Now let $k + m = i + j$, which implies $k = i + j - m$. Also from above, $k + m = i + j$ implies $i + j - 1 = k$. Hence $m = 1$. Thus $k = i + j - 1$ and hence $j \geq 2$ because $k \neq i$, which yields $m < j$. Since the trees whose indices sum to $i + j$ are generated in the order $T^{1}_{i+j-1}, \ldots, T^{i+j-1}_{1}$, the tree T^{m}_{k} exists when we need it.

Finally, f_i^j uses color i at the root of T_i^j, by construction. Since the subtrees used as descendants of the root have the form T_k^m with $k \neq i$, by induction the coloring f_i^j is proper.

Theorem 3.2. *The graph T_i^j and the coloring f_i^j have the following properties:*

a) If f' is a coloring of T_i^j different from f_i^j, then $\sum f' > \sum f_i^j$. Furthermore, if f' assigns a color different from i to the root of T_i^j, then $\sum f' - \sum f_i^j \geq j$.

b) If $j = 1$, then $\Delta(T_i^j) = 2i - 2$, achieved by the root of T_i^j. If $j \geq 2$, then $\Delta(T_i^j) = 2(i + j) - 3$.

c) The highest color used in f_i^j is $(i + j - 1)$.

Proof. a) We use induction through the order in which the trees are constructed.

Base Condition:

T_1^1 is just a single vertex, and f_1^1 gives it color 1. Conditions (a), (b), and (c) automatically follow.

Induction Hypothesis:

Properties (a), (b), and (c) hold for any tree T_k^m where $(1, 1) \le (k, m) < (i, j)$.

Induction Step:

We consider T_i^j where $(i, j) \neq (1, 1)$. For simplicity, we write T for T_i^j and f for f_i^j. To verify (a), let f' be a coloring different from f. The following two cases arise:

<u>Case 1</u>. f' assigns i to the root u of T.

In this case, f' and f differ on $T - u$. Recall that $T - u$ is the union of $2(i + j - 2)$ previously constructed trees. The colorings f' and f differ on at least one of these trees. By the induction hypothesis, the total under f' is at least the total under f on each of these $2(i + j - 2)$ subtrees, and it is larger on at least one. Hence $\sum f' > \sum f_i^j$.

<u>Case 2</u>. f' assigns a color different than i to the root u.

In this case, we need to show that $\sum f' - \sum f_i^j \ge j$. Again the induction hypothesis gives f' a total at least as large as f on each component of $T - u$. If $f'(u) \ge i + j$, then the difference $f'(u) - f(u) = (i + j) - i = j$. Hence the difference on u itself is large enough to yield $\sum f' - \sum f_i^j \ge j$. So, we may assume that $f'(u) = k$, where $1 \le k \le i + j - 1$ and $k \neq i$. Since f' is a proper coloring, it assigns a color other than k to the roots v and v' of the two copies of T_k^m in $T - u$, where $m = \lceil (i + j - k) / 2 \rceil$. Since f uses f_k^m on each copy of T_k^m, we have $f(v) = f(v') = k$. Since $f'(v)$ and $f'(v')$ differ from k, the induction hypothesis implies that on each copy of T_k^m the total of f' exceeds the total of f by at least m. Since the total is at least as large on all other components, we have

$$\sum f' - \sum f_i^j \geq k - i + 2m = k - i + 2\left\lceil \frac{i+j-k}{2} \right\rceil \geq k - i + 2\left(\frac{i+j-k}{2} \right) = j.$$

Hence part (a) of Theorem 3.2 is proved.

Proof. b) In the construction of $T = T_i^j$, we join the roots of $2(i + j - 2)$ subtrees to the root u. These subtrees have the form T_k^m for $1 \leq k \leq i - 1$ and $i + 1 \leq k \leq i + j - 1$, and always $m = \lceil (i + j - k) / 2 \rceil$. We note that $m = 1$ only when $k = i + j - 1$ or $k = i + j - 2$. By the induction hypothesis, the subtrees have maximum degree $2k - 2$ when $m = 1$, and $2(k + m) - 3$ when $m > 1$. Also note that in any case when $m \geq 1$, $2(k + m) - 3 > 2k - 2$. Thus,

$$\Delta(T_k^m) \leq 2(k + m) - 3 = 2\left(k + \left\lceil \frac{i+j-k}{2} \right\rceil \right) - 3 = 2\left(\left\lceil \frac{2k+i+j-k}{2} \right\rceil \right) - 3$$

$$= 2\left\lceil \frac{i+j+k}{2} \right\rceil - 3.$$

Also, we always have $k + m = \lceil (i + j + k) / 2 \rceil$ for the subtree T_k^m.

Case 1. $j = 1$.

We have $k \le i - 1$, and thus $\Delta(T_k^m) \le 2\lceil (i+1+k)/2 \rceil - 3 \le 2i - 3$ because maximum value that k can take is $i - 1$. Hence each vertex in $T - u$ has degree at most $(2i - 3) + 1 = 2i - 2$ in T. Since $j = 1$ and the number of subtrees is $2(i + j - 2)$, $d_T(u) = 2i - 2$ which implies that $\Delta(T) = 2i - 2$, achieved by the root.

Case 2. $j \ge 2$.

The values of k for the subtrees T_k^m now are $1 \le k \le i - 1$ and $i + 1 \le k \le i + j - 1$. By the induction hypothesis, the maximum degree of T_{i+j-1}^1 is $2(i + j - 1) - 2$ which is $2(i + j) - 4$, and is achieved by its root. In T these vertices have degree $2(i + j) - 3$, which exceeds degree of u in tree T, $d_T(u)$, which is $2(i + j) - 4$. For $k \le (i + j - 2)$, we have $\Delta(T_k^m) \le 2\lceil (i+j+k)/2 \rceil - 3 \le 2(i+j) - 5$. Hence $\Delta(T) = 2(i + j) - 3$, achieved by the roots of the trees that are isomorphic to T_{i+j-1}^1.

Proof. c) In order to prove that the maximum color used in f_i^j is $i + j - 1$, we consider two cases. First, however, we note that by the induction hypothesis, the maximum color used by f_k^m on each T_k^m within f_i^j is $k + m - 1 = \lceil (i + j + k) / 2 \rceil - 1$.

Case 1. $j = 1$.

The maximum value of k is $i - 1$, so that the maximum color used on the various T_k^m is $\lceil (i + j + k) / 2 \rceil - 1 = i - 1$. Thus our coloring f_i^j, which assigns color i to root u, has i as its maximum color and i is $(i + j - 1)$.

Case 2. $j \geq 2$.

The maximum value of k is $(i + j - 1)$, so that the maximum color used on the various T_k^m is $\lceil (i + j + k) / 2 \rceil - 1 = \lceil (2i + 2j - 1) / 2 \rceil - 1 = i + j - 1$. This color is bigger than i (the color used on u), thus the maximum color used on T_i^j is $(i + j - 1)$. This verifies (c) and completes the proof of Theorem 3.2.□

We have proved that f_i^j is the unique minimal coloring of T_i^j and that it uses $(i + j - 1)$ colors. Hence $s(T_i^j) = i + j - 1$. The maximum degree is $2i - 2$ or $2(i + j) - 3$, depending on whether $j = 1$ or $j \geq 2$. In particular, T_i^1 is a tree with strength i and maximum degree $2i - 2$.

Corollary 3.3. There exists, for each positive integer i, a tree T_i with $s(T_i) = i$ and $\Delta(T_i) = 2i - 2$.

Chapter Four

Erdös, Kubicka, and Schwenk's General Constructions

We have already seen in Chapter Two that for trees we may need arbitrarily many colors to achieve the chromatic sum, but the unique smallest tree requiring k colors is of order $O\left(2+\sqrt{2}\right)^{k}$. In other words the number of colors used in a best coloring can exceed the chromatic number by an arbitrarily large value.

In this chapter we will see that this unexpected result is not only true for trees but also for graphs with higher chromatic numbers. The work in this chapter originally appeared in [3]. We will be looking at graphs that require t colors more than their chromatic number k. We will present two different constructions.

Theorem 4.1. *For every integer $k \geq 2$ and every positive integer t, there exists a graph G_k^t, which is k-chromatic and which must use at least k + t colors to obtain its chromatic sum.*

Proof. We will construct an instance of G_k^t by using the rooted tree T_i^m, which was defined recursively in Chapter Two as follows:

T_1^1 is the trivial tree with one vertex. $T_i^m - r = \bigcup_{n=1}^{i-1}(m+i-n)T_n^1$, where r is the root of tree T_i^m. It was proved that T_i^m is the smallest tree for which in every best coloring i is forced to appear at the root and any change of that color to a lower color must increase the cost of coloring by at least m.

Construction A

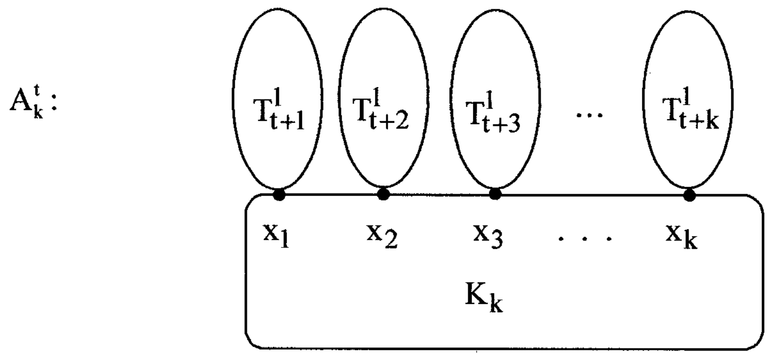

Figure 4.1. K-chromatic graph that requires t extra colors.

We obtain A_k^t by attaching at each vertex x_i of a complete graph K_k the rooted tree T_{t+i}^1. Since the best coloring of each T_{t+i}^1 requires color t + i at any root and k different colors on the vertices of K_k, the union of these colorings yields a proper coloring. This coloring is a best one for A_k^t because in any best coloring of T_{t+i}^1, t + i must appear at the root. Hence any change of color on x_i will not reduce the total cost on A_k^t. This coloring is also a best one since changing the color at the root costs at least 1 at each tree T_{t+i}^1.□

The graph A_k^t is very simple but very costly. It produces graphs of unnecessarily large order. Following the notation defined earlier, let $t_i^1 = |T_i^1|$. We know from Chapter Two that $t_i^1 = 0.25[(2+\sqrt{2})^i + (2-\sqrt{2})^i]$. Therefore the order of A_k^t is equal to

$\left|A_k^t\right| = t_{t+1}^1 + t_{t+2}^1 + \ldots + t_{t+k}^1$, and substituting the values of each t_i^1 in the equation, we obtain the following:

$$\left|A_k^t\right|$$

$$= \frac{1}{4}\left[\left(2+\sqrt{2}\right)^{t+1} + \left(2-\sqrt{2}\right)^{t+1}\right] + \frac{1}{4}\left[\left(2+\sqrt{2}\right)^{t+2} + \left(2-\sqrt{2}\right)^{t+2}\right] + \ \ldots$$

$$+\frac{1}{4}\left[\left(2+\sqrt{2}\right)^{t+k} + \left(2-\sqrt{2}\right)^{t+k}\right]$$

$$= \frac{1}{4}\left(2+\sqrt{2}\right)^{t+1}\left[1+\ldots+\left(2+\sqrt{2}\right)^{k-1}\right] + \frac{1}{4}\left(2-\sqrt{2}\right)^{t+1}\left[1+\ldots+\left(2-\sqrt{2}\right)^{k-1}\right]$$

$$= \frac{\left(2+\sqrt{2}\right)^{t+1}}{4}\left[\frac{1-\left(2+\sqrt{2}\right)^{k}}{1-\left(2+\sqrt{2}\right)}\right] + \frac{\left(2-\sqrt{2}\right)^{t+1}}{4}\left[\frac{1-\left(2-\sqrt{2}\right)^{k}}{1-\left(2-\sqrt{2}\right)}\right]$$

$$= \frac{\left(2+\sqrt{2}\right)^{t+1}}{4}\left[\left(1-\left(2+\sqrt{2}\right)^{k}\right)\left(1-\sqrt{2}\right)\right] + \frac{\left(2-\sqrt{2}\right)^{t+1}}{4}\left[\left(1-\left(2-\sqrt{2}\right)^{k}\right)\left(1+\sqrt{2}\right)\right]$$

$$= \frac{\left(2+\sqrt{2}\right)^{t}\left(2+\sqrt{2}\right)\left(1-\sqrt{2}\right)\left(1-\left(2+\sqrt{2}\right)^{k}\right)}{4} + \frac{\left(2-\sqrt{2}\right)^{t}\left(2-\sqrt{2}\right)\left(1+\sqrt{2}\right)\left(1-\left(2-\sqrt{2}\right)^{k}\right)}{4}$$

$$= \frac{(-1)\left(2+\sqrt{2}\right)^{t}\sqrt{2}\left(1-\left(2+\sqrt{2}\right)^{k}\right)}{4} + \frac{\left(2-\sqrt{2}\right)^{t}\sqrt{2}\left(1-\left(2-\sqrt{2}\right)^{k}\right)}{4}$$

$$= \frac{\sqrt{2}}{4}\left\{\left(2+\sqrt{2}\right)^{k+t} - \left(2+\sqrt{2}\right)^{t} + \left(2-\sqrt{2}\right)^{t} - \left(2-\sqrt{2}\right)^{k+t}\right\}.$$

A_4^2:

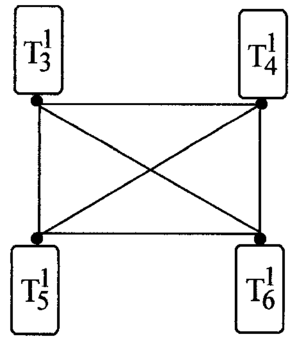

Figure 4.2. Graph of A_4^2.

For example $|A_4^2| = |T_3^1| + |T_4^1| + |T_5^1| + |T_6^1| = 10 + 34 + 116 + 396 = 556$. We notice that the order of A_k^t grows exponentially in (k + t). Therefore we want to find a better construction that produces a k-chromatic graph with strength at least k + t of a much smaller order. In fact in the following example, we are able to construct a 4-chromatic graph, B_4^2, requiring 6 colors on only 24 vertices, a 23-fold improvement on A_4^2 in the figure above. In this graph we have 10 copies of K_2 connected to one copy of K_4 as shown in Figure 4.3.

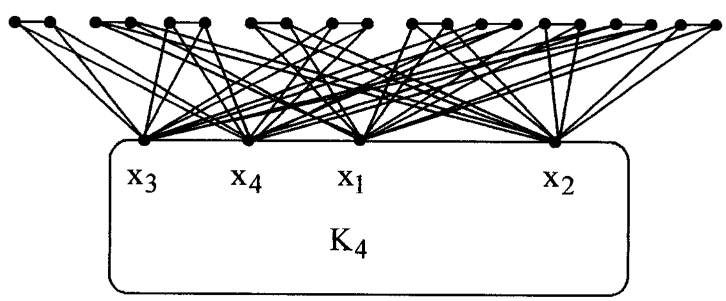

Figure 4.3. Graph of B_4^2.

Proposition 4.2. Strength of B_4^2 is 6.

Proof. We color the graph by using colors 1 and 2 on the vertices of K_2, and colors 3, 4, 5, and 6 on the four vertices of K_4 for a total cost of 10(1 + 2) + 3 + 4 + 5 + 6 = 48. Clearly the various K_2's are colored most efficiently using 1 and 2. If we use more than 6 colors, it is obvious that the cost increases. If we use less than 6 colors, then color 2 or 1 must be used on K_4.

Case 1. If 2 is used on K_4, then we save 6 – 2 = 4 on K_4, but this increases the cost of coloring K_2 by at least 5.

Case 2. If, on the other hand, 1 is used on K_4, then we save 6 – 1 = 5 on K_4, but this increases the cost of coloring K_2's by at least 10.

Case 3. If both 1 and 2 are used in K_4, then maximum saving is (6 – 1) + (5 – 2) = 8, but this costs at least 20 on the K_2's. Therefore the proposition is proved.

Construction B

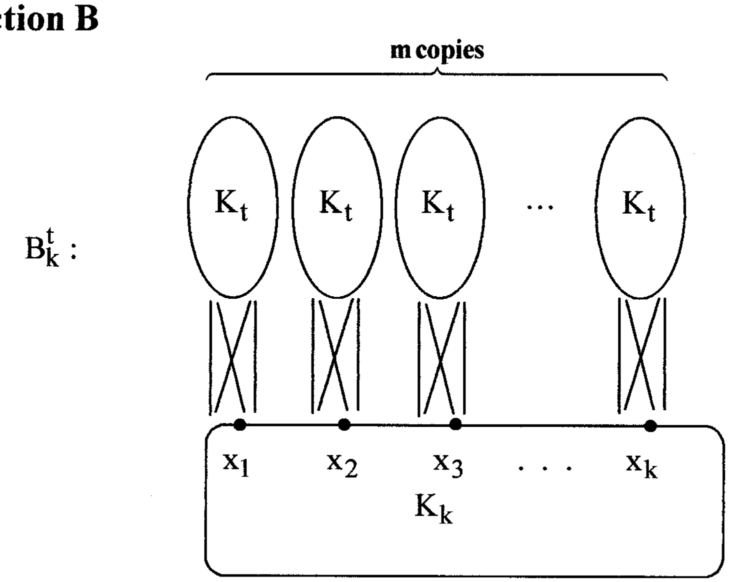

Figure 4.4. Description of construction B.

Each of the m copies of K_t in Figure 4.4 is joined to $k - t$ carefully selected vertices in K_k below, $k > t$. This forms a k-chromatic graph because t colors are available to color each copy of K_t after coloring the K_k with colors 1, 2, ..., k. For the j-th copy of K_t, we specify this selection by identifying the t vertices of K_k **not joined** to copy j. Every vertex in copy j is joined to all but vertices $x_{(j-1)t+1}$ through x_{jt} where all subscripts are taken modulo k. We show that there are values of m for which the graph B_k^t obtained by construction B requires $k + t$ colors in its best coloring and that there are small m's which yield a B_k^t of relatively small strength.

Let $\sum_i(G)$ denote the minimum possible cost of coloring the vertices of G using all colors 1 through i. When the graph G is obvious from context, we shall just write $\sum_i$. Thus we seek small values of m for which $\sum_{k+t} < \sum_{k+s}$ for all $s < t$.

<u>Definition 1</u>: Given $s \leq t$, an EKS coloring of B_k^t is a coloring which assigns 1, ..., s to vertices in each copy of K_t, $s + 1, \ldots, s + k$ to vertices in K_k, and the remaining vertices of the copies of K_t are colored as cheaply as possible.

Theorem 4.2. *For $s \leq t$, and $m(k - t) \geq k$, $\sum_{k+s}$ is achieved by an EKS coloring of B_k^t.*

Proof. Assume Theorem 4.2 is not true. Let us consider a best coloring c of B_k^t with $k + s$ colors which has the biggest possible sum of colors when restricted to just the vertices of K_k. Let $r \leq k + s$ be the biggest color not used in K_k. Then two cases arise.

Case 1. $r \le t$.

Lemma 1. $r > s$.

Proof. Let on the contrary $r \le s$. Then by definition of r, all colors from $s + 1$ to $s + k$ are used in K_k, a contradiction to our assumption.

Lemma 2. There exists a vertex x of K_k colored with some color smaller than r, say n.

Proof. From Lemma 1 we know that the biggest color r not used in K_k is greater than s. Hence, colors $r + 1$ through $k + s$ are used in K_k, but k different colors from 1 through $k + s$ have to be selected to color K_k, therefore the lemma is proved.

Since $m(k - t) \ge k$, each vertex of K_k is adjacent to at least one copy of K_t. Now consider a copy of K_t, say H, connected to x in K_k. Now every vertex of H is adjacent to x and H uses the t cheapest available colors. For H, color $r \le t$ is available, and thus H contains a vertex colored r and no vertex colored n. Now we can interchange colors r and n for the vertices v and x. If there is any vertex v_i in some other copy of K_t colored r and adjacent to x, we may also change that color to n which was not available before. Thus we can obtain an equal or cheaper coloring of B_k^t with a bigger sum of color on K_k, which gives a contradiction.

Case 2. $r > t$.

Consider a vertex v in some copy of K_t colored with r (color r has to be used somewhere) together with all its $(k - t)$ neighbors from K_k.

Subcase (a): There exist one or more vertices in copies of K_t's colored r, which have an adjacent vertex in K_k colored $n < r$ as in case 1. Interchange these colors n and r. Thus we obtain an equal cost or even cheaper coloring of B_k^t with a bigger sum of colors on K_k, which gives a contradiction.

Subcase (b): All vertices colored r have all the adjacencies in K_k colored larger than r. If there exist two such vertices colored r, we recolor one of them with a color from $\{1, \ldots, t\}$ which is not used on the K_t (since $r > t$, such a color exists). This reduces the total cost, a contradiction.

On the other hand, if there exists exactly one vertex v colored r, then there exists some color $i \leq t$ which is not used on any vertex adjacent to v. Use color i on v. If i is used on K_k, then recolor that vertex with r. This results in the same cost coloring of B_k^t and a bigger cost on K_k, a contradiction. If i is not used on K_k, then i is used on all K_t's. Find an i in K_t adjacent to a vertex colored less than r in K_k. Such a vertex in K_k exists since $m(k - t) \geq k$. Recolor that i-vertex with r. Now the cost of recoloring has not changed but we are again in subcase (a).□

Corollary 4.3. *If strength of B_k^t is $k + t$, then a minimum cost coloring uses $t + 1, \ldots, t + k$ on K_k and $1, \ldots, t$ on the various K_t's.*

Theorem 4.2 as stated in [3] does not include the condition $m(k-t) \geq k$. In the simple example of B_5^3 with $m = 2$ and $s = 2$ (i.e. $m(k-t) = 4 < 5 = k$), $\sum_{k+s}$ is not achieved by an EKS coloring. In Figure 4.5, B_5^3 (with $m = 2$) is 7-colored at a cost of 35, whereas in an EKS coloring with 7 colors, the color 1 in K_5 is replaced by the color 3 while leaving all other colors unchanged. Thus the EKS coloring of B_5^3 with 7 colors is not the most efficient.

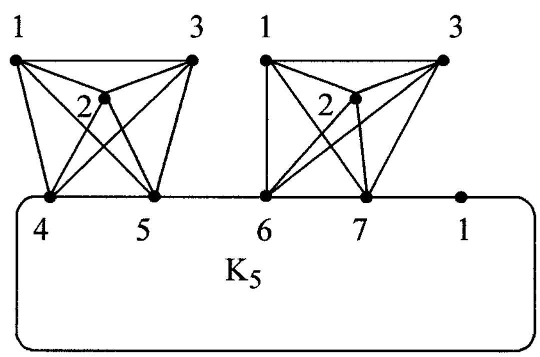

Figure 4.5. Graph of B_5^3.

In fact, contrary to the claim in [3] that $\sum_{k+s}$ is always achieved with an EKS coloring, we prove the following theorem.

Theorem 4.4. *Let $s < t < k$ and m be given such that $m(k-t) < k$, then $\sum_{k+s}$ is not achieved by an EKS coloring.*

Proof. Given an EKS coloring of B_k^t, assume $\sum_{k+s}$ is achieved by this coloring. Therefore colors $s + 1, \ldots, s + k$ are used on K_k. But since $m(k-t) < k$ some vertices $x_1, \ldots, x_r$ of K_k are not adjacent to any vertex in the copies of K_t's. If any x_i is colored with a color used more than once, we recolor it with 1. We now have a less expensive coloring, but it still uses all k + s colors, a contradiction. Otherwise each of the vertices $x_1, \ldots, x_r$ is the only vertex of its color. Now, if m > 1, one of the x_i's has color larger than (s + 1). Recolor that x_i with color 1. Since m > 1, color s + 1 is used on at least one copy of K_t. So recolor the vertex of K_k colored s + 1 by the color of x_i. This results in a cheaper coloring using all colors 1, …, k + s, a contradiction.

On the other hand, if m = 1, then clearly in the EKS coloring with total cost $\sum_{k+s}$, the s + 1 is used on x_i for some i and on the single K_t. Therefore s + 1 occurs twice and we can recolor x_i with 1, thus achieving a less costly coloring which uses all color 1, …, s + k. Hence the proof is complete.□

Our next theorem shows that there are many B_k^t graphs with strengths less than k + t . These graphs occur when m is not sufficiently large. For a simple example, consider k = 3, t = 1, m = 1. Then we have the following graph which has strength $3 < t + k = 4$.

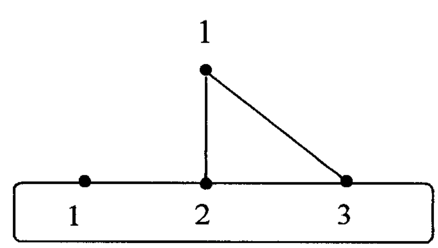

Figure 4.6. Graph of B_3^1.

Theorem 4.5. *Let $1 \leq t < k$. Then for any m such that $m(k-t) < k$, the strength of B_k^t is less than $k + t$.*

Proof. We color B_k^t as follows. Use colors 1, ..., t on each K_t. There exists $r = k - m(k-t)$ vertices of K_k which have no adjacencies in the K_t's. Color these r vertices with 1, ..., r. Color the remaining $k - r$ vertices of K_k with $t + 1, \ldots, t + k - r$.

Lemma 1. $r \leq t$.

Proof. In order to show this, assume that $r > t$. Then $k - (m(k-t)) > t$ which can be simplified to $mt - t > mk - k$, or $t(m-1) > k(m-1)$. This is a contradiction since when $m = 1$ we get $0 > 0$ and when $m > 1$, then $t > k$, which proves the lemma.

Our coloring uses each color 1, ..., r exactly $m + 1$ times. Also our coloring uses each color $r + 1, \ldots, t$ exactly m times and uses each color $t + 1, \ldots, t + k - r$ exactly once. Suppose there exists a less costly coloring using $k + t$ colors. This coloring must decrease the number of times some of the colors 1, ..., t occur. But since the $k + t$ coloring is less costly than our coloring, some of the colors among 1, ..., t must occur more frequently than in our coloring. But this is impossible, which is a contradiction to the assumption.□

Theorem 4.6. *For sufficiently large m, the graph B_k^t has strength $\geq k + t$.*

Proof. First we color B_k^t with $k + t$ colors using an EKS coloring. Suppose we change the above coloring so that we do not use color $k + t$ or any color larger than $k + t$. Suppose also that r vertices of K_k, $x_1, x_2, \ldots, x_r$, $r \geq 1$, are colored with colors less than or equal to t, thus saving at most $r(t + k - 1)$. But each x_i, $1 \leq i \leq r$, is adjacent to a large number of K_t's. Each such K_t had a vertex colored with the color now used on x_i and therefore this color has to be increased to a color larger than t. For a sufficiently large value of m the increased cost on the K_t's is greater than the saving on K_k, a contradiction.□

Theorem 4.7. *For every value of m, the strength of $B_k^t \leq k + t$.*

Proof. Color the graph B_k^t with an EKS coloring with k + t colors. Thus each copy of K_t uses colors 1, …, t and the K_k uses colors t + 1, …, t + k. This means that each color 1, …, t is used m times and each color t + 1, …, t + k is used exactly once. Suppose there is a less costly coloring c using more than k + t colors. Since c is less costly than the EKS coloring, some color from 1 to t must be used more than m times. However, by Theorem 4.5, m(k – t) ≥ k. Therefore each vertex of K_k is adjacent to at least one K_t. Hence, no color can be used more than m times. This contradiction completes the proof of the theorem.□

Corollary 4.8. *For sufficiently large m, B_k^t has strength equal to k + t.*

Proof. This follows immediately from Theorems 4.6 and 4.7.

Theorem 4.9. *Let m be an integer such that B_k^t has strength k + t. Let integer $m' > m$, and let $B_k^{t\,\prime}$ have m' copies of K_t. Then $B_k^{t\,\prime}$ also has strength k + t.*

Proof. On the contrary, assume that $B_k^{t\,\prime}$ has strength less than k + t, where coloring c′ gives minimum cost. Then apply c′ to B_k^t. Thus we have a color less than or equal to t on a vertex x of K_k, and a color greater than t on some copies of K_t adjacent to x. Let c be a coloring of B_k^t which yields a minimum cost. By the assumption, c must use k + t colors. Also, by Corollary 4.3, c uses colors t + 1, ..., t + k on K_k and $c(B_k^t) < c'(B_k^t)$. Now extend c to $B_k^{t'}$ by coloring each additional copy of K_t with 1, ..., t. Clearly c' applied to these $m' - m$ extra copies of K_t cannot cost less than the cost of c applied to these K_t's, thus c applied to $B_k^{t'}$ is cheaper than c' applied to $B_k^{t'}$, a contradiction. Hence the theorem is proved.□

Now we show that for certain values of m, $\sum_{k+t} < \sum_{k+s}$ $\forall$ s < t. Thus, using Theorem 4.7 for these values of m, graph B_k^t has strength (t + k). Furthermore, by Theorem 4.9, once a value of m is determined for which B_k^t has strength t + k, each larger value of m corresponds to a B_k^t with strength t + k. We know that $1 + 2 + 3 + \ldots + t = \binom{t+1}{2}$.

Thus $\sum_{k+t} = m(1 + 2 + \ldots + t) + ((t+1) + (t+2) + \ldots + (t+k))$

$$= m\binom{t+1}{2} + kt + \binom{k+1}{2}.$$

When using only k + s colors, the sum is

$$\sum_{k+s} = m(1 + 2 + \ldots + s) + L_s + ((s+1) + (s+2) + \ldots + (s+k))$$

$= m\binom{s+1}{2} + L_s + ks + \binom{k+1}{2}$. Here L_s is the cheapest possible sum of colors over m(t – s) vertices of the copies of K_t when in every copy the first s vertices use color 1 through s, and the vertices of K_k use colors s + 1 through s + k. We want to show that for certain values of m, the coloring c_s (which uses k + s colors) is more costly than the coloring c_t (which uses k + t colors), s < t. The difference of the cost of coloring K_k with c_t and the cost of coloring K_k with c_s is kt – ks = k(t – s). Therefore, for the total cost to be larger using c_s rather than c_t, the increase of the sum of colors over the m(t – s) vertices of the various K_t's (called D_s), must be bigger than k(t – s). Therefore, we need to show that for certain m, $D_s > (t-s)k$, and thus we will have $\sum\limits_{k+t} < \sum\limits_{k+s}$.

Let G_s denote the graph obtained from B_k^t by deleting s vertices from every copy of K_t. Notice that the coloring c_s, where the colors 1 through s are assigned to the removed vertices, can be transformed to a best coloring c_s' of G_s by diminishing every color by s. We will show that for certain values of m to be determined, $\sum_{k+t} < \sum_{k+t-1}$. Let v_i be a vertex in the i^{th} copy of K_t which is colored t by c_t and colored $c_{t-1}(v_i)$ by the coloring c_{t-1}. Thus when we summate the difference between the colors assigned to v_i in c_{t-1} and the color t assigned in c_t over all m copies of K_t, we get $D_{t-1} = \sum\limits_{i=1}^{m}(c_{t-1}(v_i) - t) > k$, since we are assuming $\sum_{k+t} < \sum_{k+t-1}$. Consider now the graph G_{t-1}.

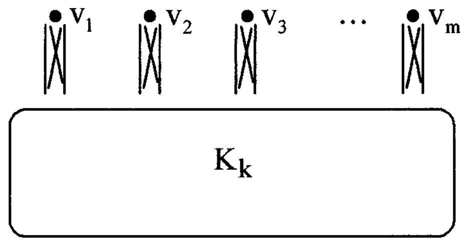

Figure 4.7. Graph of G_{t-1}.

Subtracting the value (t–1) from every $c_{t-1}(v_i)$, we obtain $c_{t-1}(v_i) = c'_{t-1}(v_i) + (t-1)$ which yields $D_{t-1} = \sum_{i=1}^{m}[c'_{t-1}(v_i) - (t-(t-1))] > k$. This implies that $\sum_{i=1}^{m} c'_{t-1}(v_i) > k + m$.

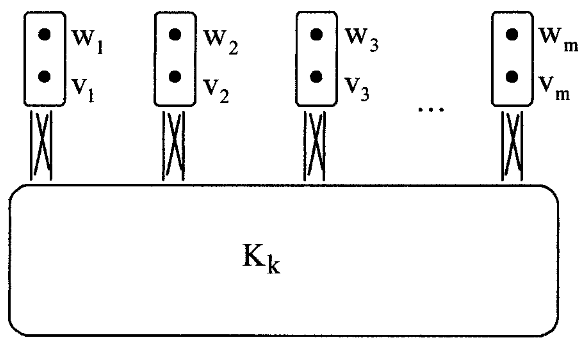

Figure 4.8. Graph of G_{t-2}.

Let w_i be the vertex in the i^{th} copy of K_t which is colored $(t-1)$ by c_t and colored $c_{t-2}(w_i)$ by coloring c_{t-2}. For the graph G_{t-2}, in every best coloring, we cannot color w_i cheaper than using the color of v_i increased by one. Hence, $\sum_{i=1}^{m} c'_{t-2}(v_i) > k + m$ and $\sum_{i=1}^{m} c'_{t-2}(w_i) > k + m + m$. Therefore, we obtain

$$\sum_{i=1}^{m}[c'_{t-2}(v_i) + c'_{t-2}(w_i)] > (k+m) + (k + (2m)) \quad \text{.....................................(4.1)}$$

The term 2m appears because the color on w_i is at least one more than that on v_i. Thus, in the graph B_k^t we obtain $D_{t-2} = \sum_{i=1}^{m}[c_{t-2}(v_i) + c_{t-2}(w_i) - t - (t-1)]$

$$= \sum_{i=1}^{m}[c'_{t-2}(v_i) + c'_{t-2}(w_i) + 2(t-2) - (2t-1)]$$

$$= \sum_{i=1}^{m}[c'_{t-2}(v_i) + c'_{t-2}(w_i)] - 3m > 2k.$$

The second equality above follows from the equations $c_{t-2}(v_i) - (t-2) = c'_{t-2}(v_i)$ and $c_{t-2}(w_i) - (t-2) = c'_{t-2}(w_i)$, and then the inequality follows from equation 4.1 above. Similarly, for any $s < t$ we obtain $D_s > (t - s)k$. Hence, in order to verify that B_k^t has strength $k + t$, it is enough to show that for our certain m the inequality $\sum_{i=1}^{m} c_{t-1}(v_i) - mt > k$ holds.

Theorem 4.10. *If m and t are positive integers such that $mt \leq k$ and $m > \frac{1+\sqrt{1+8k}}{2}$, then the strength of B_k^t is $k + t$ and $t < \frac{\sqrt{1+8k}}{4} - \frac{1}{4}$.*

Before proving the theorem for some fixed k, we consider if such an m and t can exist. Note $t < -\frac{1}{4} + \frac{\sqrt{1+8k}}{4} \approx \frac{1}{2}\sqrt{2k}$ and $m > \frac{1+\sqrt{1+8k}}{2} \approx \sqrt{2k}$. For example, when $k = 200$ and $m \geq 21$, then $0 < t < 10$. Thus there are many values of m and t such that $mt \leq k$.

Proof. Recall that the 1^{st} copy of K_t is not adjacent to $x_1, ..., x_t$,

the 2^{nd} copy of K_t is not adjacent to $x_{t+1}, ..., x_{2t}$,

the 3^{rd} copy of K_t is not adjacent to $x_{2t+1}, ..., x_{3t}$,

$\vdots \quad \vdots \quad \vdots \quad \vdots$

the i^{th} copy of K_t is not adjacent to $x_{(i-1)t+1}, ..., x_{it}$,

$\vdots \quad \vdots \quad \vdots \quad \vdots$

the m^{th} copy of K_t is not adjacent to $x_{(m-1)t+1}, ..., x_{mt}$.

Since $mt \leq k$, each vertex x_i $(1 \leq i \leq mt)$ is not adjacent to exactly one copy of K_t, and each vertex x_i, $i > mt$, is adjacent to all the copies of K_t. Thus, in coloring c_{t-1}, the v_i's (as described in Figure 4.7) in the various copies of K_t have to be assigned different colors and have to be colored with the cheapest colors available. Now, according to the coloring c_{t-1}, we color vertices in each copy of K_t with colors 1 to t – 1, and then color the k vertices in K_k with colors t to t + k – 1. Therefore, the vertices v_i in the various copies of K_t can be colored cheaply with colors t to (t + m – 1) such that

$$\sum_{i=1}^{m} c_{t-1}(v_i) = t + (t+1) + \ ... \ + (t+m-1) = mt + \binom{m}{2} = mt + \frac{m(m-1)}{2}.$$

As mentioned immediately preceding the statement of this theorem, we want a value of m such that $\sum_{i=1}^{m} c_{t-1}(v_i) > k + mt$, hence $\sum_{i=1}^{m} c_{t-1}(v_i) = mt + \frac{m(m-1)}{2} > k + mt$,

which simplifies to $\frac{m^2 - m}{2} > k$, or $m^2 - m > 2k$.

Now we need to show that $m^2 - m > 2k$ or $m^2 - m - 2k > 0$.

Notice that our hypothesis is $m > \dfrac{1+\sqrt{1+8k}}{2}$

$$\Rightarrow \left[\left(m - \frac{1}{2}\right) - \frac{\sqrt{1+8k}}{2}\right] > 0.$$

Therefore, $\left[\left(m-\dfrac{1}{2}\right)-\dfrac{\sqrt{1+8k}}{2}\right]\left[\left(m-\dfrac{1}{2}\right)+\dfrac{\sqrt{1+8k}}{2}\right]>0$

$$\Rightarrow \left(m-\frac{1}{2}\right)^2 - \left(\frac{\sqrt{1+8k}}{2}\right)^2 > 0$$

$$\Rightarrow m^2 - m + \frac{1}{4} - \left(\frac{1+8k}{4}\right) > 0$$

$\Rightarrow \quad m^2 - m - 2k > 0$ as required.

Now, since $m > \dfrac{1+\sqrt{1+8k}}{2}$ and $k \geq mt$, we obtain the following inequality:

$$k \geq mt > \frac{t}{2} + \sqrt{\frac{t^2}{4} + 2kt^2}$$

or, $k > \dfrac{t}{2} + \sqrt{\dfrac{t^2}{4} + 2kt^2}$

or, $\left(k - \dfrac{t}{2}\right)^2 > \dfrac{t^2}{4} + 2kt^2$

or, $2kt^2 + kt - k^2 < 0$

or, $\left[t - \left(\dfrac{-1+\sqrt{1+8k}}{4}\right)\right]\left[t - \left(\dfrac{-1-\sqrt{1+8k}}{4}\right)\right] < 0.$

Since $t > 0$, we obtain $t < -\frac{1}{4} + \frac{\sqrt{1+8k}}{4}$.

Hence the theorem is proved.□

Theorem 4.11. *If $m \geq \frac{k(k+1)}{k-t}$ then B_k^t has strength $k + t$.*

Proof. Since $\frac{k}{k-t} > 1$, it follows that $m \geq \frac{k(k+1)}{k-t} > k+1 > k$, and therefore $mt > k$. Thus, we have one or more vertices of K_k non-adjacent to at least two of the K_t's.

By the discussion between Theorems 4.9 and 4.10, in order to prove B_k^t has strength $k + t$, it suffices to show that $\sum_{k+t} < \sum_{k+t-1}$. Now, suppose on the contrary, the strength of B_k^t is less than $k + t$. Let the vertices of each copy of K_t be colored from 1 to $t - 1$ leaving one vertex uncolored in each K_t. Color the vertices of K_k from t to $t + k - 1$. We will now color the uncolored vertices in K_t as cheaply as possible. Let v_i's be the single uncolored vertices left in K_t's, and x_j's be the vertices in K_k. Each v_i is adjacent to $(k - t)$ x_j's by the construction of B_k^t. There are m v_i's, therefore the number of edges from the v_i's to the x_j's is $m(k - t) \geq \left(\frac{k(k+1)}{k-t}\right)(k-t) = k(k+1)$. Also, there are k x_j's, so each x_j on average is adjacent to at least $k + 1$ v_i's.

Lemma 1. Each x_j is adjacent to at least $k + 1$ v_i's.

Proof. Assume the contrary. Then some x_j is adjacent to at most k v_i's and some other x_j is adjacent to at least $k + 2$ v_i's. But from the definition of B_k^t (the difference between the number of adjacencies of any two x_j's to the vertices in K_t can at most be 1), this cannot occur. Hence, Lemma 1 is proved.

Recolor the x_j's starting from t to t + k – 1. The v_i's, at least k + 1 in number, adjacent to the x_j colored t have to be colored t + 1. Although we save a cost of k on K_k, we increase the cost on the v_i's by k + 1, a net increase of at least 1. Therefore the strength is k + t and the theorem is proved.□

Order

All A_k^t graphs have strength k + t. We have already seen at the beginning of this chapter that the order of A_k^t grows exponentially in (k + t) because it is equal to

$$|A_k^t| = \frac{\sqrt{2}}{4}\left[\left(\sqrt{2}+2\right)^{k+t} - \left(\sqrt{2}+2\right)^{t} - \left(2-\sqrt{2}\right)^{k+t} - \left(2-\sqrt{2}\right)^{t}\right].$$

On the other hand, the order of B_k^t is $|B_k^t| = k + mt$, which clearly depends on m. When m is sufficiently small, the graphs B_k^t have strengths less than k + t. From Corollary 4.8, however, we know that there are infinitely many B_k^t's with strength k + t. Each such B_k^t has order k + mt. So we consider B_k^t of small order i.e. with m being relatively small. By Theorem 4.11, if $m = \left\lceil \frac{k(k+1)}{k-t} \right\rceil$, then B_k^t has strength k + t. Furthermore k – t ≥ 1, therefore $m = \left\lceil \frac{k(k+1)}{k-t} \right\rceil \leq k(k+1)$. Hence, the order of B_k^t with this value of m is $|B_k^t| = k+mt \le k+(k(k+1))t = k + k^2t + kt < k+k^2+k^3$. Thus, the order of these B_k^t's is only a cubic in k. Hence, in terms of order, the construction B_k^t is smaller than the construction A_k^t.

Strength

The chromatic number of A_k^t, $\chi(A_k^t)$ is k, and the strength of A_k^t is k + t, where there is no restriction on t. Therefore $\frac{\text{strength}(A_k^t)}{\chi(A_k^t)}$ has no bound. In other words, the strength of A_k^t can be made arbitrarily larger than the chromatic number $\chi(A_k^t)$. In case of construction B_k^t, $\chi(B_k^t) = k$ and strength of $B_k^t \le k + t$ where $t + k < k + k = 2k$. Hence, the strength of $B_k^t < 2\ \chi(B_k^t)$, which can be expressed as $\frac{\text{strength}(B_k^t)}{\chi(B_k^t)} < 2$. In other words, the strength of B_k^t can never be as large as twice its chromatic number. We notice that although B_k^t uses fewer vertices than A_k^t to achieve a strength of k + t, the A_k^t construction works for all $t \ge 1$, where as in the construction B_k^t, t must be less than k.

this value of m is $\left|B_k^t\right| = k + mt \leq k + \left(k\left(k+1\right)\right)t = k + k^2t + kt < k + k^2 + k^3$. Thus, the order of these B_k^t's is only a cubic in k. Hence, in terms of order, the construction B_k^t is smaller than the construction A_k^t.

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The chromatic number of A_k^t, $\chi(A_k^t)$ is k, and the strength of A_k^t is k + t, where there is no restriction on t. Therefore $\frac{\text{strength}(A_k^t)}{\chi(A_k^t)}$ has no bound. In other words, the strength of A_k^t can be made arbitrarily larger than the chromatic number $\chi(A_k^t)$. In case of construction B_k^t, $\chi(B_k^t) = k$ and strength of $B_k^t \leq k + t$ where $t + k < k + k = 2k$. Hence, the strength of $B_k^t < 2\ \chi(B_k^t)$, which can be expressed as $\frac{\text{strength}(B_k^t)}{\chi(B_k^t)} < 2$. In other words, the strength of B_k^t can never be as large as twice its chromatic number. We notice that although B_k^t uses fewer vertices than A_k^t to achieve a strength of k + t, the A_k^t construction works for all $t \geq 1$, where as in the construction B_k^t, t must be less than k.